JOURNAL OF FUNCTIONAL ANALYSIS 38, 315–323 (1980)

An Invariant Subspace Theorem

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Communicated by the Editors

Received November 3, 1978; revised December 11, 1979

In this paper it is proved that every operator on a complex Hilbert space whose spectrum is a spectral set has a nontrivial invariant subspace.

1. INTRODUCTION

If $\mathscr{H}$ is a complex, separable, Hilbert space and T is a bounded operator on $\mathscr{H}$, then a compact subset of $\mathbb{C}$ is a *spectral set* for T if

$$\|f(T)\| \leqslant \max\{|f(z)|: z \in K\}$$

for every rational function f with poles off K. In [10] von Neumann introduced the notion of spectral set and showed that if T has $\|T\| = 1$, then the closed unit disc, $\mathbb{D}^-$, is a spectral set for T. For this reason any operator T whose spectrum is a spectral set for T is called a *von Neumann operator*. Hence, if $\|T\| = 1$ and $\sigma(T) = \mathbb{D}^-$, then T is a von Neumann operator. If T or T^* is a subnormal operator, then T is a von Neumann operator. In this paper we prove:

THEOREM. *If T is a von Neumann operator, then T has a nontrivial invariant subspace.*

This theorem generalizes the recent result of Scott Brown that every subnormal operator has a nontrivial invariant subspace, although the proof relies heavily on Brown's techniques. We wish here to thank him for an early manuscript [3] containing his results.

Finally, we remark that the results of this paper were outlined in [1].

2. PRELIMINARIES

$\mathscr{H}$ is complex, separable, infinite dimensional Hilbert space with inner product $(\cdot, \cdot)$; $S_1 = \{x \in \mathscr{H}: \|x\| = 1\}$. $\mathscr{L}(\mathscr{H})$ denotes the bounded linear transformations on $\mathscr{H}$. For $T \in \mathscr{L}(\mathscr{H})$, $\sigma(T)$ denotes the spectrum of T and $\sigma_{ap}(T)$ denotes the approximate point spectrum. If K is a compact subset of $\mathbb{C}$, the complex

0022-1236/80/120315-09$02.00/0

numbers, then K is a *spectral set* for T if $\sigma(T) \subset K$ and $\|f(T)\| \leqslant \max\{|f(z)|: z \in K\}$ for all rational functions f with poles off K. For K a compact subset of $\mathbb{C}$, $R(K)$ denotes the uniform closure of the rational functions with poles off K in $C(K)$, the continuous funjtions on K. $R(K)$ is said to be *Dirichlet* (more precisely: a Dirichlet algebra on ∂K) if $\operatorname{Re}(R(K))$ is dense $C_{\mathbb{R}}(\partial K)$; i.e., the real parts of the functions in $R(K)$, when restricted to ∂K, are dense in the continuous real valued functions on ∂K. If $R(K)$ is Dirichlet and $z \in K^0$ (K^0 is the interior of K), then there exists a unique measure λ_z with spt $\lambda_z \subset \partial K$ and $\int f d\lambda_z = f(z)$ for all $f \in R(K)$. Let $K^0 = \bigcup_{n=1}^{\infty} G_n$ be an enumeration of the components of K^0 and fix $z_n \in G_n$. Set $m = \sum_{n=1}^{\infty} 2^{-n}\lambda_{z_n}$. This m will be referred to in the sequel as *harmonic measure*. If $R(K)$ is Dirichlet, then $H^\infty(\partial K)$ is the closure of $R(K)$ in the weak-star (hereafter simply, w*) topology of $L^\infty(m)$. For G open in $\mathbb{C}$, $H^\infty(G)$ denotes the functions bounded and holomorphic on G. A standard result from potential theory is that the natural map from $R(K)$ into $H^\infty(K^0)$ (the one that sends $f \in R(K)$ to $f \in H^\infty(K^0)$) extends to an isometry from $H^\infty(\partial K)$ onto $H^\infty(K^0)$. (See [9].)

The following four lemmas from function theory will be used frequently in the sequel.

LEMMA A. *If $R(K)$ is Dirichlet, then the components of K^0 are simply connected.*

LEMMA B. *If $R(K)$ is Dirichlet and $J \subset K$ with J compact, and each component of K/J reaches out to ∂K, then $R(J)$ is a Dirichlet algebra.*

LEMMA C. *If K_n is a decreasing sequence of compact sets such that $R(K_n)$ is Dirichlet for each n, then $R(\bigcap_{n=1}^{\infty} K_n)$ is Dirichlet.*

LEMMA D. *If $R(K)$ is Dirichlet and μ is a finite positive measure on ∂K which is singular with respect to harmonic measure m on ∂K, $h \in H^\infty(\partial K)$, and $g \in L^\infty(\mu)$, then there is a sequence $\{r_n\}$ in $R(K)$ with $r_n \to h$ w* in $H^\infty(\partial K)$, $r_n \to g$ w* in $L^\infty(\mu)$, and* $\sup_n \| r_n \|_K \leqslant \max(\| h \|_{H^\infty(\partial K)}, \| g \|_\mu)$.

Lemma A is well known; Lemmas B, C, and D are all due to Sarason (in [9] see Theorem 1, Lemma 7.1, and Lemma 4.3, respectively). A direct statement and proof of Lemma B is Corollary 33.5 of Gamelin's most recent (unpublished) notes on approximation theory. Also see [5].

The following seems to be a useful notion in the study of von Neumann operators.

DEFINITION. If $T \in \mathscr{L}(\mathscr{H})$, a compact subset $K \subset \mathbb{C}$ is *D-spectral* for T if K is a spectral set for T and $R(K)$ is a Dirichlet algebra.

The following theorem is due independently to Lautzenheiser [6] and Mlak [8].

LAUTZENHEISER–MLAK DECOMPOSITION THEOREM. *Let K be a spectral set for $T \in \mathscr{L}(\mathscr{H})$ and let G_1, G_2,..., be the nontrivial Gleason parts for $R(K)$. Then*

$$T = N \oplus \left(\sum_{i=1}^{\infty} \oplus T_i\right),$$

where

(i) *N is normal and $\sigma(N) \subset \partial K$, and*

(ii) *$\bar{G}_i$ is a spectral set for T_i (in particular, $\sigma(T_i) \subset \bar{G}_i$).*

Lemma 3.1 of [6] says that T_i in the above theorem is not trivial if $G_i^0 \cap \sigma(T) \neq \square$. Combining this observation with the well-known fact that for $R(K)$ Dirichlet the nontrivial Gleason parts of $R(K)$ are the components of K^0 yields the following lemma.

LEMMA E. *If K is D-spectral for $T \in \mathscr{L}(\mathscr{H})$, and K^0 has more than one component which meets $\sigma(T)$, then T has a nontrivial reducing subspace.*

A final result related to function theory that will be needed is:

LEMMA F. *If K is D-spectral for $T \in \mathscr{L}(\mathscr{H})$, and $K^0 = \square$, then T is normal.*

Proof. Since $R(K)$ is Dirichlet every point in $\partial K = K$ is a peak point. Bishop's peak point criterion thus implies $R(K) = C(K)$. A theorem of von Neumann [10] says that if K is a spectral set for T and $R(K) = C(K)$ then T is normal. Q.E.D.

We now discuss some duality results for $\mathscr{L}(\mathscr{H})$. Let $\mathscr{C}_1$ denote the operators on $\mathscr{H}$ of finite trace. $L(\mathscr{H}) = \mathscr{C}_1^*$ with the action, $\langle C, A\rangle = \text{trace}\,(AC)$, if $\mathscr{C}_1$ is given the trace norm. We refer to the *weak-star topology* $\mathscr{L}(\mathscr{H})$ inherits as a dual as the *w^* topology*. Some authors choose to call this topology "ultraweak," though it is stronger than what is commonly referred to as the "weak operator topology." If K is a compact set with $\sigma(T) \subset K$, then $\mathscr{R}_K$ will denote the w^* closure of the rational functions in T with poles off K.

3. A WEAK FUNCTIONAL CALCULUS

Let $T \in \mathscr{L}(\mathscr{H})$ and suppose K is a spectral set for T. In this section we extend the $R(K)$ functional calculus to a wider class of functions. An alternative approach to the one taken here could be developed by using the dilation theorem of Foias [4]. See also [7]. We make no attempt to describe the strongest possible results since they will not be needed subsequently in this paper.

If K is a spectral set for $T \in \mathscr{L}(\mathscr{H})$ and f is a rational function with poles off K, then clearly $f(T)$ is well defined. If Φ_K denotes the map that sends f to $f(T)$, then the fact that K is a spectral set says that Φ extends to a norm contraction $\Phi_K : R(K) \to \mathscr{R}_K$. Now suppose that $R(K)$ is Dirichlet and fix $x \in S_1 = \{x \in \mathscr{H} : \|x\| = 1\}$. The map $L_x(f) = (f(T)x, x)$ is easily seen to be a linear functional of norm one on $R(K)$. Hence, there exists a unique measure μ_x with the properties,

(1) $\operatorname{spt} \mu_x \subset \partial K$;

(2) $\mu_x \geqslant 0, \quad \|\mu_x\| = 1$;

(3) for all $f \in R(K)$, $(f(T)x, x) = \int f \, d\mu_x$.

Now, let m be a harmonic measure for $R(K)$ and let $d\mu_x = f_x \, dm + dv_x$, where $f_x \in L^1(m)$ and $dv_x \perp dm$. For a sequence $\{x_n\}_{n=1}^{\infty} \subset S_1$, dense in S_1, let $\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_{x_n}$ and let $v = \sum_{n=1}^{\infty} 2^{-n} v_{x_n}$. Evidently, μ is a probability measure; v, f_x, and v_x are non-negative; and v is singular with respect to harmonic measure. Note that all these measures depend on the choice of K and that μ and v depend on the choice of $\{x_n\}_{n=1}^{\infty}$, facts that often will be suppressed in the following discussion.

PROPOSITION 1. *If $T \in \mathscr{L}(\mathscr{H})$ and K is D-spectral for T then Φ_K extends to a norm contractive algebra homomorphism Φ_K: $H^{\infty}(\partial K) \oplus L^{\infty}(v) \to \mathscr{R}_K$. Furthermore Φ_K is continuous when domain and range have their* w* *topologies.*

Proof. Let $h \in H^{\infty}(\partial K)$ and $g \in L^{\infty}(v)$. By Lemma D there exists a sequence $\{r_p\}_{p=1}^{\infty}$, $r_p \in R(K)$ for all p, such that:

(1) $r_p \to h$ w* in $H^{\infty}(\partial K)$;

(2) $r_p \to g$ w* in $L^{\infty}(v)$;

(3) $\sup \|r_p\|_K \leqslant \max(\|h\|_m, \|g\|_v)$.

Recall that $v = \sum_{n=1}^{\infty} 2^{-n} v_{x_n}$, where $\{x_n\}_{n=1}$ is a sequence dense in S_1. A simple computation shows that (1) and (2) imply that $\langle A, r_p(T)\rangle$ converges for all $A = \sum_{n=1}^{\infty} \lambda_n (x_n \oplus x_n)$, where $\{\lambda_n\}$ is chosen so that $\lambda_n \geqslant 0$ for all n and $\sum \lambda_n < \infty$. Since linear combinations of A's of this form are dense in the trace class and since (3) implies $\sup \|r_p(T)\| < \infty$, we conclude that $\{r_p(T)\}$ converges w*. Set $\Phi_K(h \oplus g) = \lim_{p\to\infty} r_p(T)$. By repeating the above argument Φ_K is seen to be w* sequentially continuous so that (Krein–Smulian theorem) Φ_K is w* continuous. Also clear is that range $\Phi_K \subset \mathscr{R}_K$ and that Φ_K is an algebra homomorphism. There remains to show that Φ_K is a norm contraction. But

$$\begin{aligned} \|\Phi_K(h \oplus g)\| &\leqslant \lim \|r_n(T)\| \\ &\leqslant \lim \|r_n\|_K \\ &\leqslant \max(\|h\|_m, \|g\|_v) \\ &= \|h \oplus g\|. \end{aligned}$$

Q.E.D.

Observe that Φ_K depends not only on T and K but also on the choice of dense sequence $\{x_n\}_{n=1}^{\infty}$.

4. Proof of the Theorem

The basic idea of the proof is that condition (R) (below) forces the functional calculus of Prosition 1 to be as "nice" for von Neumann operators as it is for subnormal operators (see Prosition 2). Once we have established this fact we link up with the ideas of Brown to finish the proof. Throughout this section T will be a fixed von Neumann operator. Clearly, (since otherwise T has a nontrivial invariant subspace) we may assume:

(R) T has no nontrivial reducing subspaces.

Lemma 1. *For any K, D-spectral for T, and any $x \in S_1$, $v_x = 0$.*

Proof. This is a consequence of (R). Suppose not; i.e., there exists an $x \in \mathscr{H}$, $\| x \| = 1$ and $v_x > 0$. Pick a sequence $\{x_n\} \subset S_1$ with $\{x_n\}$ dense in S_1 and $x_1 = x$. Since the Φ_K of Proposition 1 is a homomorphism, $\Phi_K(0 \oplus 1)$ is a projection commuting with T (an idempotent of norm $\leqslant 1$ is a projection). Since $(\Phi_K(0 \oplus 1)x, x) = \int 1\, dv_x > 0$, condition (R) forces $\Phi_K(0 \oplus 1) = I$, the identity operator on $\mathscr{H}$. Thus, $\mu_{x_n} \perp m$ for all n. If $v_x = v_{x_i}$ for all $j \neq i$, then T is a multiple of the identity. So assume $v_{x_1} \neq v_{x_2}$. It follows that there is a characteristic function χ in $L^\infty(v)$ with the properties (a) $0 < \int \chi\, dv_{x_1}$ and (b) $\int \chi\, dv_{x_2} < 1$. Then $\Phi_K(0 \oplus \chi)$ is a projection commuting with T, not 0 by (a) and not I by (b). This contradicts (R). Q.E.D.

In light of Lemma 1 it is easy to see that if condition (R) is in force, then Φ_K of Proposition 1 does not depend on the choice of sequence $\{x_n\}_{n=1}^{\infty}$. In a certain sense, Φ_K does not even depend on K. For suppose that $K_1 \subset K_2$ and $\phi \in H^\infty(\partial K_2)$. Then in particular, $\phi \in H^\infty(K_2{}^0)$ and hence, by restriction, $\phi \in H^\infty(K_1{}^0)$. Evidently then, $\phi \in H^\infty(\partial K_1)$. An easily verified fact is that $\Phi_{K_1}(\phi) = \Phi_{K_2}(\phi)$. For these reasons, if K is D-spectral for T and $\phi \in H^\infty(\partial K)$, we set $\phi(T) = \Phi_K(\phi \oplus 0)$, suppressing all dependence of Φ_K on K and $\{x_n\}_{n=1}^{\infty}$.

Lemma 2. *If K is D-spectral for T, K^0 has one component, and ϕ is a conformal map from K^0 onto $\mathbb{D}$, then $\phi(T)$ has no nontrivial reducing subspaces.*

Proof. First note that $\phi \in H^\infty(\partial K)$ so that $\phi(T)$ is defined by Proposition 1. Observe that $\phi^{-1} \in H^\infty(\mathbb{D})$, so that there exists a bounded sequence of polynomials $\{p_j\}$ with $p_j \to \phi^{-1}$ w^* in H $(\partial\mathbb{D})$. This implies $\lim_{n\to\infty}(p_j \circ \phi)(z) = z$ for every $z \in K^0$, which in turn implies, since $\{\| p_j \circ \phi \|\}$ is uniformly bounded, that $p_j \circ \phi \to z$ w* in $H^\infty(\partial K)$. Hence by Proposition 1, $(p_j \circ \phi)(T) \to T$ w* in $\mathscr{R}_K$. Now note that $p_j(\phi(T)) = (p_j \circ \phi)(T)$. What this proves is that T is in the w* closed algebra generated by the polynomials in $\phi(T)$. Evidently if $\phi(T)$ commutes with a projection, then so does T. Thus, condition (R) implies that $\phi(T)$ has no nontrivial reducing subspaces. Q.E.D.

LEMMA 3. *If K is D-spectral for T, K^0 has one component, and ϕ is a conformal map from K^0 onto $\mathbb{D}$, then*

$$\sigma(\phi(T)) \cap \mathbb{D} = \phi(\sigma T) \cap K^0).$$

Proof. Let $\lambda \in K^0$. Since ϕ is conformal, $\phi(z) - \phi(\lambda) = (z - \lambda)h(z)$, where both h and $1/h$ are in $H^\infty(K^0)$. The functional calculus gives $\phi(T) - \phi(\lambda) = (T - \lambda)h(T)$, where $h(T)$ is invertible. Hence $\phi(T) - \phi(\lambda)$ is invertible if and only if $T - \lambda$ is invertible. Q.E.D.

The author wishes to thank the referee for pointing out this simple proof of Lemma 3.

We now are ready to prove our central lemma.

LEMMA 4. *There exists a set K with the following properties*:

(P1) *K is D-spectral for T,*

(P2) *K^0 has one component, and*

(P3) *If ϕ is a conformal map from K^0 onto $\mathbb{D}$, then $\partial\mathbb{D} \subset \sigma(\phi(T))$.*

Proof. For each countable ordinal α, define, via transfinite induction, a compact set K_α as follows:

$$K_1 = [\sigma(T)]^\wedge, \qquad \text{the polynomially convex hull of } \sigma(T);$$

$$K_{\alpha+1} = K_\alpha \backslash (\phi_\alpha^{-1}(V_\alpha) \cup U_\alpha);$$

$$\text{if } \alpha \text{ is a limit ordinal, then } K_\alpha = \bigcap_{\beta<\alpha} K_\beta .$$

Here, U_α is the union of the components, G, of $K_\alpha{}^0$ that have the property that $G \cap \sigma(T) = \square$. By a transfinite argument (see below), $R(K_\alpha)$ is a Dirichlet algebra. It follows that if $L_\alpha = K_\alpha \backslash U_\alpha$, $R(L_\alpha)$ is Dirichlet (Lemma B). Also, by transfinite induction (see below), $\sigma(T) \subset L_\alpha$. Thus L_α is D-spectral for T. Evidently, by Lemma E (since, by definition of L_α, *all* the components of $L_\alpha{}^0$ meet $\sigma(T)$) and condition (R), $L_\alpha{}^0$ has one component. This component, by Lemma A, is simply connected. ϕ_α is a conformal map from $L_\alpha{}^0$ onto $\mathbb{D}$. V_α is the union of the components, U, of $\mathbb{D} \backslash \phi_\alpha(\sigma(T) \cap L_\alpha{}^0)$ with the property that $\partial U \cap \partial\mathbb{D}$ contains a nontrivial arc I, with the property that $I \cap \sigma(\phi_\alpha(T)) = \square$.

To summarize what we need to prove to make sense of the definition:

(i) $R(K_\alpha)$ is a Dirichlet algebra for every α, and

(ii) $\sigma(T) \subset K_\alpha$ for every α.

That (i) and (ii) hold for $\alpha = 1$ is clear. Suppose (i) and (ii) hold for all $\beta < \alpha$, where α is a limit ordinal. Since $\beta_1 < \beta_2$ implies $K_{\beta_2} \subset K_{\beta_1}$, $K_\alpha = \bigcap_{n=1}^\infty K_{\beta_n}$ and K_{β_n} is decreasing for appropriately chosen ordinals $\beta_n < \alpha$. Hence Lemma C and the inductive assumption imply $R(K_\alpha)$ is Dirichlet. That $\sigma(T) \subset K_\alpha$ is also clear. Now suppose (i) and (ii) hold for some α. Then by Lemma B, $R(L_\alpha)$ is Dirichlet. Since each component of V_α meets $\partial\mathbb{D}$, by the maximum principle each component of $\phi_\alpha^{-1}(V_\alpha)$ meets $\partial L_\alpha \subset \partial K_\alpha$. By Lemma B, $R(K_{\alpha+1})$ is Dirichlet. To see that $\sigma(T) \subset K_{\alpha+1}$, just note that $U_\alpha \cap \sigma(T) = \square$ by definition of U_α and that $\phi_\alpha^{-1}(V_\alpha) \cap \sigma(T) = \square$ by Lemma 3.

Thus the definition of the sets K_α makes sense. Let $\gamma =$ first countable ordinal such that $K_\gamma = K_{\gamma+1}$. γ exists since $\{\alpha \mid \alpha$ is countable and $K_\alpha = K_{\alpha+1}\} \neq \square$ (otherwise there is an uncountable family of disjoint nonempty open sets). Set $K = K_\gamma$. By the argument justifying the definition of the sets K_α, we know that (P1) holds. Since $K_\gamma = K_{\gamma+1}$, $U_\alpha = \square$, which implies that every component of K^0 meets $\sigma(T)$. If K_α has fewer than one component (i.e., $K^0 = \square$), then Lemma F implies T is normal, contradicting (R). If K^0 has more than one component, then we contradict (R) via Lemma E. Thus K^0 has precisely one component, which is (P2). Finally the fact that $V_\gamma = \square$ is easily seen to imply (P3). Q.E.D.

Throughout the rest of this paper K and ϕ are as in Lemma 4. For $h \in H^\infty(\partial K)$, let

$$\| h \|_{K^0 \cap \sigma(T)} = \sup\{| h(z)| : z \in K^0 \cap \sigma(T)\}.$$

LEMMA 5. *$\| h \|_{H^\infty(\partial K)} = \| h \|_{K^0 \cap \sigma(T)}$ for all $h \in H^\infty(\partial K)$.*

Proof. Suppose not. Then there is an $h \in H^\infty(\partial K)$ with $\| h \|_{H^\infty(\partial K)} > \| h \|_{K^0 \cap (T)}$. Using Lemma 3, it follows that $\| h \circ \phi^{-1} \|_D > \| h \circ \phi^{-1} \|_{\sigma(\phi(T)) \cap \mathbb{D}}$, where $h \circ \phi^{-1} \in H^\infty(\mathbb{D})$. Now, using the idea of the construction in Lemma 3.1 of [3], we obtain a smooth arc $\gamma \subset \mathbb{D}$ with endpoints $z_1, z_2 \in \partial\mathbb{D}$, $z_1 \neq z_2$, which separates $\mathbb{D}$ into two parts, the arc γ missing $\sigma(\phi(T)) \cap \mathbb{D}$. Let V be an open set in $\mathbb{D}$ with the properties that $\gamma \cap \mathbb{D} \subset V$, $V^- \cap \partial\mathbb{D} = \{z_1, z_2,\}$, $V^- \cap \sigma(\phi(T)) = \{z_1, z_2\}$, and V and V^- are simply connected. Set $L = K\backslash\phi^{-1}(V)$. By Lemma B and Lemma 3, L is a D-spectral set for T. Since L^0 contains two components it then follows, by Lemma E and condition (R), that one of the components of L^0 misses $\phi(T)$. Hence, by Lemma 3, one of the components determined by γ misses $\sigma(\phi(T))$. Since $\partial\mathbb{D} \subset \sigma(\phi(T))$ (Lemma 4) we conclude that there exists $z \in \partial\mathbb{D}$ and $\epsilon > 0$ such that

$$\Delta(z, \epsilon) \cap \sigma(\phi(T)) = \partial\mathbb{D} \cap \Delta(z, \epsilon), \tag{*}$$

where

$$\Delta(z, \epsilon) = \{w \in \mathbb{C} : | w - z | < \epsilon\}.$$

We now show that $z \notin \sigma(\phi(T))$, thus contradicting (P3) in Lemma 4. Let $J = K\backslash\phi^{-1}(\mathbb{D} \cap \Delta(z, \epsilon))$. By Lemma B and (*), J is D-spectral for T. For clarity we revert momentarily to our original notation for the functional calculus. Note first that $\phi \in H^\infty(\partial L)$ and that $\Phi_J(\phi) = \Phi_K(\phi)$. Also observe that $1/(\lambda - \phi) \in H^\infty(\partial J)$ for $\lambda \in \mathbb{D} \cap \Delta(z, \epsilon)$. It follows that

$$(\lambda - \Phi_K(\phi))\, \Phi_{J(1/(\lambda-\phi))} = \Phi_J(\lambda - \phi)\, \Phi_{J(1/(\lambda-\phi))} = I,$$

the identity on $\mathscr{H}$. Similarly,

$$\Phi_{J(1/(\lambda-\phi))(\lambda-\Phi_K(\phi))} = I.$$

We conclude that $(\lambda - \Phi_K(\phi))^{-1} = \Phi_{J(1/(\lambda-\phi))}$. Thus we see that if $|z - \lambda| \leqslant \epsilon/2$,

$$\begin{aligned}
\| 1/(\lambda - \phi(T))\| &= \| \Phi_{J(1/(\lambda-\phi))}\| \\
&\leqslant \left\| \frac{1}{\lambda - \phi} \right\|_{H^\infty(\partial J)} \\
&= \sup_{w \in J^0} \left| \frac{1}{\lambda - \phi(w)} \right| \\
&= \sup \left\{ \left| \frac{1}{\lambda - \eta} \right| : \eta \in \mathbb{D} \backslash \Delta(z, \epsilon)^- \right\} \\
&\leqslant 2/\epsilon.
\end{aligned}$$

Thus $z \notin \sigma(\phi(T))$ and, with this contradiction (to (P3) in Lemma 4, the proof of Lemma 5 is complete. Q.E.D.

Proposition 2. *The Φ of proposition 1 associated with the K of Lemma 4 is a norm isometric,* w* *homeomorphic, algebra isomorphism from $H^\infty(\partial K)$ onto $\mathscr{R}_K$.*

Proof. Fix $h \in H^\infty(\partial K)$. By Proposition 1, $\| h \| \geqslant \| h(T)\|$. For $\lambda \in K^0 \cap \sigma(T)$, Lemma 3 implies that

$$\| h(T)\| \geqslant | h(\lambda)| .$$

Hence

$$\| h(T)\| \geqslant \sup_{\lambda \in K^0 \cap \sigma(T)} | h(\lambda)| = \| h \|_{K^0 \cap \sigma(T)} .$$

But, by Lemma 5, $\| h \|_{K^0 \cap \sigma(T)} = \| h \|$. Hence $\| h(T)\| \geqslant \| h \| \geqslant \| h(T) \|$ and we conclude that Φ is an isometry. Now, by Proposition 1, Φ is w* continuous. Since $1/(\lambda - T) \in$ range Φ for each $\lambda \notin K$, the proof will be complete if range Φ is w* closed (Φ will be w* open since it will be the adjoint of an invertible map). Since Φ is an isometry, (ball $\mathscr{R}_K) \cap \Phi(H^\infty(\partial K)) = \Phi$ (ball $H^\infty(\partial K)$). Since ball $H^\infty(\partial K)$ is w* compact it follows that (ball $\mathscr{R}_K) \cap \Phi(H^\infty(\partial K))$ is w* compact. Hence, by the Krein–Smulian theorem, $\Phi(H^\infty(\partial K))$ is w* closed. Q.E.D.

We can now conclude the proof by use of Theorem 4.1 in [2].

THEOREM. *Let $A \in \mathscr{L}(\mathscr{H})$ with $\|A\| = 1$ and*

$$\sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |h(\lambda)| = \|h\|_\infty$$

for all $h \in H^\infty(\mathbb{D})$. Then A has a nontrivial invariant subspace.

If K and ϕ are as in Proposition 2 and $A = \phi(T)$, then Lemma 3 and Proposition 2 together imply that

$$\sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |h(\lambda)| = \|h\|_\infty$$

for all $h \in H^\infty(\mathbb{D})$. Since also $\|\phi(T)\| \leqslant 1$, by the above theorem, $\phi(T)$ has an invariant subspace. Since T is a w* limit of polynomials in $\phi(T)$ (see proof of Lemma 2 this paper) it follows that T has an invariant subspace.

ACKNOWLEDGMENTS

The author wishes to thank Scott Brown for pointing out a gap in his original proof which made it necessary to cite Theorem 4.1 of [2]. I also want to thank Tom Bagby and Glenn Schober for useful conversations concerning this paper. It was Jim Thomson who impressed upon me the depth of Roger Lautzenheiser's thesis and Donald Sarason's paper [9] from which the ideas of this paper are gleaned. There is not enough time or space to express my gratitude to John Conway.

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